Exercise: Derive the rule of thumb for the wavelength of a free electron after passing through an accelerating voltage U from the De Broglie relation. Calculate examples.

Detailed solution and remarks for students

Louis De Broglie arrived at the wavelength $\lambda = h/p$ of the (massless) photon by initially "speculatively" equating the energy from the frequency with the energy from the momentum. This relation was quickly extended to quantum particles with mass (free electron, i.e. an electron not attached to an ion, atom, or molecule and is free to move under the influence of an applied electric or magnetic field; photoelectron; Auger electron; electron in solids and fluids; protons; ions and neutral particles in general). Most practical measurements in electron microscopy are carried out on moving free electrons.

De Broglie's hypothesis is justified because the values obtained for the wavelengths are in excellent agreement with the experimentally measured wavelengths of the particles.

***Fundamental physical constants*:**

$h = 6.6260702 \cdot 10^{-34}$ J·s	Planck's constant,
$m_e = 9.1093837 \cdot 10^{-31}$ kg	*rest mass* or *invariant mass of a (stationary) electron,* (For U > 100 kV, the rest mass must be relativistically corrected by multiplication with the Lorentz Factor ==> *relativistic mass*
$q_e = 1.6021766 \cdot 10^{-19}$ C	elementary charge of an electron
$c = 2.99792 \cdot 10^{8}$ m/s	speed of light

$E_0 = \mathrm{m_e \cdot c^2} = 0.5110$ MeV $= 8.187 \cdot 10^{-14}$ J energy of the electron at rest (rest energy)

Variables *(*NB: variables *in italics*, while units are written in straight letters*)*

U acceleration voltage or potential difference to ground

$p_e = m_e \cdot v = m_e \cdot \sqrt{2E_{kin}/m_e} = \sqrt{2m_e \cdot E_{kin}} = \sqrt{2\, m_e \cdot q_e \cdot U}$

momentum of an electron accelerated by U.

The potential energy of the electric tension, $E_{pot} = q \cdot U$, is released as kinetic energy E_{kin}

Useful conversions *for the following exercise*

Joule 1 J = 1 N m = 1 kg m²/s²
Coulomb 1 C = 1 J/V

Hence:

$$\lambda = \frac{h}{p_e} = \frac{h}{\sqrt{2\, m_e \cdot q_e \cdot U}}$$

$$\approx \frac{6.6 \cdot 10^{-34}\ \mathrm{J \cdot s}}{\sqrt{2 \cdot 9.1 \cdot 10^{-31}\ \mathrm{kg} \cdot 1.6 \cdot 10^{-19}\ \mathrm{C} \cdot U}}$$

$$= \frac{6.6 \cdot 10^{-34}}{\sqrt{2 \cdot 9.1 \cdot 10^{-31} \cdot 1.6 \cdot 10^{-19} \cdot U}} \cdot \frac{\mathrm{J \cdot s}}{\sqrt{\mathrm{kg \cdot C}}}$$

$$\approx \frac{1.223 \cdot 10^{-9}}{\sqrt{U}} \cdot \frac{1}{\sqrt{\mathrm{kg} \cdot \frac{\mathrm{J}}{\mathrm{V}} \cdot \frac{1}{(\mathrm{J \cdot s})^2}}}$$

$$= \frac{1.223 \cdot 10^{-9}}{\sqrt{U}} \cdot \frac{1}{\sqrt{\frac{\text{kg}}{\text{V} \cdot \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{s}^2}}}$$

$$\approx \frac{\sqrt{1.5} \cdot 10^{-9}}{\sqrt{U}} \cdot \sqrt{\text{V}} \cdot \text{m}$$

$$= \sqrt{1.5 \cdot \frac{\text{V}}{U}} \text{ nm}$$

This equation gives good approximate values for the electron wavelength for acceleration voltages below about 100 kV.

Examples: $U = 10$ kV $\quad \lambda = \sqrt{\frac{1.5 \cdot \text{V}}{10 \text{ kV}}} \text{ nm} = \sqrt{\frac{1.5 \cdot \cancel{\text{V}}}{10{,}000 \cancel{\text{V}}}} \text{ nm} \approx 0.012 \text{ nm}$

$U = 20$ kV $\quad \lambda \approx 0.009$ nm

$U = 30$ kV $\quad \lambda \approx 0.007$ nm

$U = 40$ kV $\quad \lambda \approx 0.006$ nm

Empirical formulas such as

$$\lambda \approx \sqrt{\frac{1.5}{\hat{U}}} \text{ nm} \qquad \text{“with } \hat{U} = |U/\text{V}|\text{”}$$

and $\lambda \approx \sqrt{\frac{1.5}{U}}$ nm $\qquad$ “*measure the accelerating voltage U in volts*”

with a *legend*, indication the dimensions in brackets, or similar are outdated. They do not comply with the standard ISO notation.

The formula $\lambda \approx \sqrt{\frac{1{,}5}{U}}$ nm is incorrect. There is a mismatch between the dimension on the left-hand side [length] to that on the right-hand side [length / $\sqrt{\text{V}}$] of the equation.

For **voltages higher than about 100 kV**, the momentum of the electron must be corrected for **relativistic effects.** It is derived from the total energy, E_{total} , of the particle.

$$E_{total} = \sqrt{{E_0}^2 + c^2 \cdot p^2}$$

$$p = \frac{\sqrt{{E_{total}}^2 - {E_0}^2}}{c} = \frac{\sqrt{\left({E_0}^2 + {E_{kin}}^2\right) - {E_0}^2}}{c} = \frac{\sqrt{2 \cdot E_0 \cdot E_{kin} + {E_{kin}}^2}}{c}$$

$$\Rightarrow \lambda = \frac{h\,c}{\sqrt{2 \cdot E_0 \cdot E_{kin} + {E_{kin}}^2}} = \frac{hc}{\sqrt{2 \cdot m_e \cdot c^2 \cdot q_e \cdot U_a + (q_e \cdot U_a)^2}}$$

$$= \frac{h/c}{\sqrt{2 \cdot m_e \cdot q_e \cdot U_a \cdot (1 + \frac{q_e \cdot U_a}{2 \cdot m_e \cdot c^2})}}$$

In the non-relativistic De Broglie relation above, it is sufficient to multiply the rest mass in p_e by the Lorentz factor γ. This leads to a somewhat more complicated formula for the electron's wavelength:

$$\lambda \approx \sqrt{\frac{1.5}{\frac{U}{\mathrm{V}} \cdot \left(1 + 0.983 \cdot 10^{-6} \cdot \frac{U}{\mathrm{V}}\right)}}\ \mathrm{nm} \approx \sqrt{\frac{1.5}{\frac{U}{V} \cdot \left(1 + 10^{-6} \cdot \frac{U}{\mathrm{V}}\right)}}\,\mathrm{nm}$$

The electron rest mass (symbol: m_e) is the mass of a *stationary electron*, also known as the invariant mass of the electron. It is one of the fundamental constants of physics. This quantity m_e is frame invariant and velocity independent. However, some texts group the Lorentz factor with the mass factor to define a new quantity called the *relativistic mass*, $m_{\mathrm{relativistic}} = \gamma m_e$. This quantity is evidently velocity dependent, and from it arises the notion that "mass increases with speed".